Journal of Engineering Mathematics 18 (1984) 259–271.

Surface waves on a homogeneously fluidized bed

D.J. NEEDHAM *

Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, England

(Received February 2, 1984)

Summary

This paper examines the small-amplitude free-surface motions on a homogeneously fluidized bed for which the density ratio $\rho_f/\rho_s \ll 1$, where ρ_f is the fluid density and ρ_s is the solid density. It is shown that to leading order in ρ_f/ρ_s, the surface motions are independent of the dynamics of the bed varying only with the dimensions of the bed and the conditions at uniform fluidization. Also, at leading order, it is found that provided the fluidization remains homogeneous there is a direct analogy between waves on the free surface of a fluidized bed and waves on the free surface of an inviscid liquid (when surface tension is neglected).

1. Introduction

Fluidization is a process in which a bed of solid particles, whose diameters range typically from 1 mm down to 10^{-2} mm, is subject to a vertical, upward flow of fluid. On increasing the fluid flow speed, a point is reached where the upward drag exerted by the fluid on the particles balances the downward gravitational force acting on the particles, which then become buoyant. At this point the bed is said to be fluidized. Upon fluidization, a clearly defined free surface separates the fluidized region from the pure fluid region above. This free surface is defined by the uppermost layer of particles. The fluidized region exhibits many large-scale phenomena, such as buoyancy, which are similar to those of a liquid. In particular when the fluidized bed is disturbed, waves appear on the free surface showing many similarities to the waves on the free surface of a liquid in a bounded tank. A brief description of the surface waves on a fluidized bed is given by Gelperin and Einstein [2]. Rice and Wilhelm [7] used two models to describe the motion of the free surface of a fluidized bed, although it is not clear how their results relate to a fluidized bed since neither of the models uses equations of motion derived from the elementary balances in both the fluid and particle phases in the fluidized region. In fact, in both models the interface is treated as that which separates two immiscible liquids, whereas the interface separating a fluidized region from the pure fluid region is in fact permeable to the fluid phase, and defined by the uppermost layer of particles.

In this paper we consider the possible small-amplitude surface motions on a bounded fluidized bed in which compressibility effects in the fluid phase are negligible, and $\rho_f/\rho_s \ll 1$ where ρ_f is the fluid density and ρ_s is the particle density. In the fluidized region,

* Present address: University of East Anglia, School of Mathematics and Physics, Norwich NR4 7TJ, England.

the fundamental continuum equations of motion for two-phase flows are used while in the pure fluid region the usual continuity and Euler equations are used. Since we are considering essentially gas-fluidized beds, the effect of viscosity in the fluid phase can be neglected. Also, the effect of particle phase viscosity is neglected, although we are able to assess the small contribution of this term once the solution to the inviscid problem has been determined.

2. Equations of motion

The equations of motion in the fluidized region for fluidized beds in which $\rho_f/\rho_s \ll 1$ and fluid and particle phase viscosities have been neglected are, after neglecting terms of $O(\rho_f/\rho_s)$ following Murray [4,5],

$$\frac{\partial E}{\partial t} + \mathrm{div}(E\mathbf{u}) = 0, \tag{2.1}$$

$$-\frac{\partial E}{\partial t} + \mathrm{div}([1-E]\mathbf{v}) = 0, \tag{2.2}$$

$$\rho_s(1-E)\left\{\frac{\partial \mathbf{v}}{\partial t} + [\mathbf{v}\cdot\nabla]\mathbf{v}\right\} = \beta(\mathbf{u}-\mathbf{v}) - (1-E)\rho_s g\mathbf{k} - \nabla P_s, \tag{2.3}$$

$$\nabla P = -\beta(\mathbf{u}-\mathbf{v}), \tag{2.4}$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{v}$ is the particle velocity, E is the voidage, P is the fluid phase pressure, ρ_s is the particle density and $\mathbf{k}$ is a unit vector directed vertically upwards; β is the drag coefficient per unit bed volume and P_s is the particle phase pressure, both of which depend on E, and the equations are closed once this functional dependence is proposed. Equations (2.1) and (2.2) express conservation of mass in the fluid and particle phases respectively, while equations (2.3) and (2.4) are momentum balances in the particle and fluid phases respectively. Detailed derivations of the equations of motion governing a fluidized bed are given by Murray [4,5] and Anderson and Jackson [1], in which the physical significance of each term is discussed at some length.

In the pure fluid region, the equations of motion, after neglecting the effects of viscosity, are the usual continuity and Euler equations; namely,

$$\mathrm{div}(\mathbf{q}) = 0, \tag{2.5}$$

$$\rho_f\left\{\frac{\partial \mathbf{q}}{\partial t} + [\mathbf{q}\cdot\nabla]\mathbf{q}\right\} = -\nabla P_l - \rho_f g\mathbf{k}, \tag{2.6}$$

where $\mathbf{q}$ is the fluid velocity and P_l is the fluid pressure in the pure fluid region, and ρ_f is the fluid density.

3. Boundary conditions

In a fluidized bed there are three different types of boundary; the side walls which are impermeable to both fluid and particle phases, the bottom grid which allows fluid to pass but is impermeable to the particle phase, and the free surface being the interface between the fluidized and pure fluid regions (as defined by the uppermost layer of particles) allows fluid to pass.

Since we are considering an inviscid model, we must apply the condition of zero normal fluid and particle velocities at the side walls. At the bottom of the bed the normal particle velocity must be zero, while the mass flux of fluid across the bottom must be continuous. Finally, at the free surface the normal particle velocity must equal the normal rate of change of the interface, the mass flux of fluid across the interface must be continuous; and the normal stress across the interface must be continuous.

The equations of motion (2.1)–(2.6) and the above boundary conditions together with suitable initial conditions determine completely the flow in the fluidized region, the free surface and the flow in the pure fluid region above. To proceed further the geometry of the fluidized bed must be considered. In this paper we examine the two-dimensional surface waves on a fluidized bed of rectangular cross-section and axisymmetric surface waves on a fluidized bed of cylindrical cross section, although it should be noted that three-dimensional surface waves on a rectangular fluidized bed may be treated in a similar manner. We consider first the case of two-dimensional surface waves on a fluidized bed of rectangular cross-section.

4. Fluidized bed of rectangular cross-section

We introduce Cartesian coordinates (x, y, z), where z measures distance vertically upwards and x and y are perpendicular axes measuring distance in the horizontal plane. The velocity vectors may now be written as $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{q} = (q_x, q_y, q_z)$. As we are considering only two-dimensional flow, we take $\partial/\partial y \equiv 0$ and $v_y = u_y = q_y \equiv 0$.

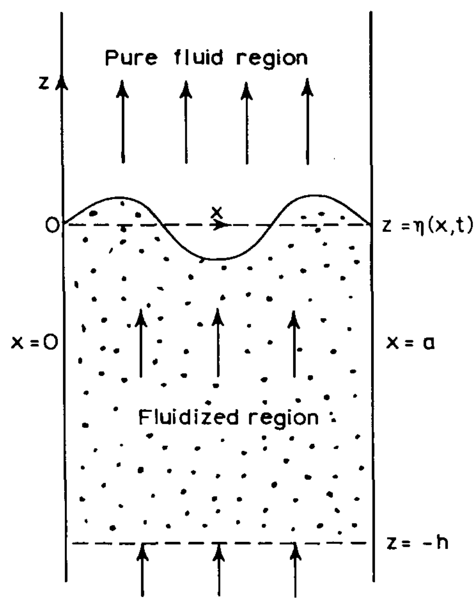

Figure 1. The coordinate system for the fluidized bed of rectangular cross-section.

The bottom of the bed is fixed at $z = -h$ and the free surface at $z = \eta(x, t)$, while the side walls are given by $x = 0$ and $x = a$. The coordinate system is shown in Fig. 1.

The boundary conditions may now be formulated in terms of the dependent variables. In the fluidized region, $-h \leqslant z \leqslant \eta(x, t)$, we have

$$v_x = u_x = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = a, \tag{4.1}$$

$$Eu_z - U_T = 0 \quad \text{on} \quad z = -h, \tag{4.2}$$

$$v_z = 0 \quad \text{on} \quad z = -h. \tag{4.3}$$

In the pure fluid region, $\eta(x, t) \leq z < \infty$, we have

$$q_x = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = a, \tag{4.4}$$

$$\mathbf{q} \quad \text{remains bounded as} \quad z \to \infty. \tag{4.5}$$

Finally, at the free surface $z = \eta(x, t)$ we have,

$$P + P_s - P_l = 0, \tag{4.6}$$

$$\frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} - v_z = 0, \tag{4.7}$$

$$(Eu_x - q_x + (1 - E)v_x)\frac{\partial \eta}{\partial x} - (Eu_z - q_z + (1 - E)v_z) = 0. \tag{4.8}$$

The simplest solution of equations (2.1)–(2.6) which satisfies the boundary conditions (4.1)–(4.8) is that in which the bed is uniformly fluidized of depth h. The free surface is horizontal and is fixed at $z = 0$, that is

$$z = \eta(x, t) = 0. \tag{4.9}$$

In the fluidized region, $-h \leq z \leq 0$,

$$\mathbf{u} = u_0\mathbf{k},\ \mathbf{v} = \mathbf{0},\ E = \epsilon_0,\ P = P_0(z) = P_{0l} - g(1 - \epsilon_0)\rho_s z - P_s(\epsilon_0) \tag{4.10}$$

where $u_0 = (1 - \epsilon_0)\rho_s g/\beta_0$, $\beta_0 = \beta(\epsilon_0)$ and P_{0l} is the pressure at the interface. Also, if M is the total mass of particles per unit cross-section, then $h = M/(1 - \epsilon_0)\rho_s a$. In the pure fluid region, $0 < z < \infty$, we have

$$\mathbf{q} = \epsilon_0 u_0 \mathbf{k},$$

$$P_l = P_{l0}(z) = P_{0l} - \rho_f g z. \tag{4.11}$$

The uniform solution (4.9)–(4.11) is illustrated in Fig. 2. This uniform state is now used to introduce the following dimensionless quantities

$$\mathbf{u} = u_0\mathbf{u}',\ \mathbf{v} = u_0\mathbf{v}',\ \mathbf{q} = u_0\mathbf{q}',\ x = hx',\ z = hz',$$

$$t = (h/u_0)t',\ P = \rho_s u_0^2 P',\ P_s = \rho_s u_0^2 P_s',\ P_l = \rho_f u_0^2 P_l',$$

$$\beta = (\rho_s u_0/h)\beta',\ \eta = h\eta'.$$

Substituting into the equations (2.1)–(2.6) and on dropping primes for convenience we obtain the following set of dimensionless equations:

$$\frac{\partial E}{\partial t} + \nabla \cdot E\mathbf{u} = 0,$$

$$-\frac{\partial E}{\partial t} + \nabla \cdot (1-E)\mathbf{v} = 0,$$

$$(1-E)\left\{\frac{\partial \mathbf{v}}{\partial t} + [\mathbf{v} \cdot \nabla]\mathbf{v}\right\} = \beta(\mathbf{u}-\mathbf{v}) - \frac{1-E}{F}\mathbf{k} - \nabla P_s, \tag{4.12}$$

$$\nabla P = -\beta(\mathbf{u}-\mathbf{v})$$

in the region $-1 \le z \le \eta(x, t)$, and

$$\nabla \cdot \mathbf{q} = 0,$$
$$\frac{\partial \mathbf{q}}{\partial t} + [\mathbf{q} \cdot \nabla]\mathbf{q} = -\nabla P_l - \frac{g}{F}\mathbf{k} \tag{4.13}$$

in the region $\eta(x, t) \le z < \infty$, where $F = u_0^2/gh$ is the Froude number. In terms of the dimensionless quantities the boundary conditions (4.1)–(4.8) become

$$\begin{aligned}
&v_x = u_x = 0 && \text{on} \quad x = 0 \quad \text{and} \quad x = \sigma,\\
&Eu_z - u_T = 0 && \text{on} \quad z = -1,\\
&v_z = 0 && \text{on} \quad z = -1,\\
&q_x = 0 && \text{on} \quad x = 0 \quad \text{and} \quad x = \sigma,\\
&P + P_s = 0 && \text{on} \quad z = \eta(x, t),\\
&\frac{\partial n}{\partial t} + v_x\frac{\partial \eta}{\partial x} - v_z = 0 && \text{on} \quad z = \eta(x, t),
\end{aligned} \tag{4.14}$$

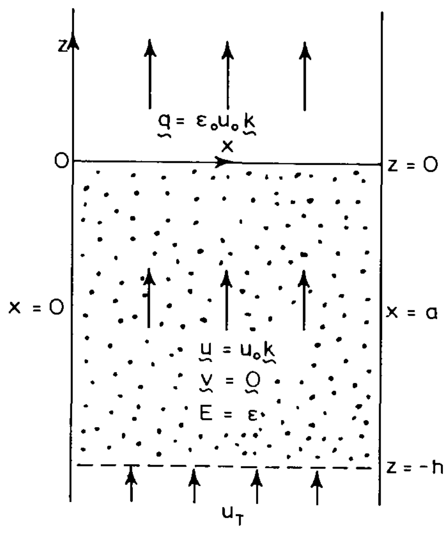

Figure 2. The uniform state.

$$(Eu_x - q_x + (1-E)v_z)\frac{\partial\eta}{\partial x} - (Eu_z - q_z + (1-E)v_z) = 0 \quad \text{on} \quad z = \eta(x,t)$$

where $\sigma = a/h$ and terms of $O(\rho_f/\rho_s)$ arising in the dimensionless boundary conditions have been neglected, to be consistent with neglecting such terms in the equations of motion in the fluidized region.

In terms of the dimensionless quantities, the uniform state may now be written

$$\begin{aligned} &\mathbf{u} = \mathbf{k},\ \mathbf{v} = \mathbf{0},\ E = \epsilon_0, \\ &P = P_0(z) = P'_{0l} - \frac{1-\epsilon_0}{F}z - P_s(\epsilon_0) \end{aligned} \tag{4.15}$$

for $-1 \le z \le 0$ and

$$\begin{aligned} &\mathbf{q} = \epsilon_0\mathbf{k}, \\ &P_l = P_{l0}(z) = P'_{0l} - z/F \end{aligned} \tag{4.16}$$

where again terms of $O(\rho_f/\rho_s)$ have been neglected. Here $\beta_0 = (1-\epsilon_0)/F$ and $P'_{0l} = P_{0l}/\rho_f u_0^2$.

We now examine the possible surface motions due to small perturbations about the uniform state (4.15) and (4.16) when the bed is homogeneously fluidized. Perturbation quantities $\bar{\mathbf{u}}$, $\bar{E}$, $\bar{v}$, $\bar{P}$, $\bar{\mathbf{q}}$, $\bar{P}_l$ are introduced, and we write

$$\begin{aligned} &\mathbf{u} = \mathbf{k} + \alpha\bar{\mathbf{u}},\ \mathbf{v} = \alpha\bar{\mathbf{v}},\ E = \epsilon_0 + \alpha\bar{E},\ P = P_0(z) + \alpha\bar{P},\ \mathbf{q} = \epsilon_0\mathbf{k} + \alpha\bar{\mathbf{q}}, \\ &P_l = P_{l0}(z) + \alpha\bar{P}_l,\ \eta = \alpha\bar{\eta} \quad \text{where} \quad |\alpha| \ll 1. \end{aligned} \tag{4.17}$$

Substitution of (4.17) into the equations of motion (4.12) and (4.13), and boundary conditions (4.14) and retaining terms of $O(\alpha)$ only, results in a set of linear partial differential equations for the perturbed quantities subject to the linearized boundary conditions; namely,

$$\frac{\partial\bar{E}}{\partial t} + \frac{\partial\bar{E}}{\partial z} + \epsilon_0\nabla\cdot\bar{\mathbf{u}} = 0, \tag{4.18}$$

$$-\frac{\partial\bar{E}}{\partial t} + (1-\epsilon_0)\nabla\cdot\bar{\mathbf{v}} = 0, \tag{4.19}$$

$$(1-\epsilon_0)\frac{\partial\bar{\mathbf{v}}}{\partial t} = \frac{1-\epsilon_0}{F}(\bar{\mathbf{u}} - \bar{\mathbf{v}}) + \left(\beta'_0 + \frac{1}{F}\right)\bar{E}\mathbf{k} - \left.\frac{dP_s}{dE}\right|_{\epsilon_0}\nabla\bar{E}, \tag{4.20}$$

$$\nabla\bar{P} = -\frac{1-\epsilon_0}{F}(\bar{\mathbf{u}} - \bar{\mathbf{v}}) - \beta'_0\bar{E}\mathbf{k} \tag{4.21}$$

in the fluidized region, where $\beta'_0 = d\beta/dE|_{\epsilon_0}$, and

$$\nabla\cdot\bar{\mathbf{q}} = 0, \tag{4.22}$$

$$\frac{\partial\bar{\mathbf{q}}}{\partial t} + \epsilon_0\frac{\partial\mathbf{q}}{\partial z} = -\nabla\bar{P}_l \tag{4.23}$$

in the pure fluid region, subject to the boundary conditions

$$\bar{v}_x = \bar{u}_x = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = \sigma, \tag{4.24}$$

$$\epsilon_0 \bar{u}_z + \bar{E} = 0 \quad \text{on} \quad z = -1, \tag{4.25}$$

$$\bar{v}_z = 0 \quad \text{on} \quad z = -1, \tag{4.26}$$

$$\bar{q}_x = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = \sigma, \tag{4.27}$$

$$\bar{\mathbf{q}} \quad \text{remains bounded as} \quad z \to \infty, \tag{4.28}$$

$$\bar{P} + \left.\frac{dP_s}{dE}\right|_{\epsilon_0} \bar{E} = \frac{1-\epsilon_0}{F}\bar{\eta} \quad \text{on} \quad z = 0, \tag{4.29}$$

$$\frac{\partial \bar{\eta}}{\partial t} = \bar{v}_z \quad \text{on} \quad z = 0, \tag{4.30}$$

$$\epsilon_0 \bar{u}_z + \bar{E} - \bar{q}_z + (1-\epsilon_0)\bar{v}_z = 0 \quad \text{on} \quad z = 0. \tag{4.31}$$

Eliminating between equations (4.18)–(4.21), a single equation for $\bar{E}$ may be obtained, namely,

$$\frac{\partial^2 \bar{E}}{\partial t^2} + \frac{1}{\epsilon_0 F}\frac{\partial \bar{E}}{\partial t} + \frac{(1-\epsilon_0) - \epsilon_0(F\beta_0' + 1)}{\epsilon_0 F}\frac{\partial \bar{E}}{\partial z} + \left.\frac{dP_s}{dE}\right|_{\epsilon_0} \nabla^2 \bar{E} = 0. \tag{4.32}$$

Examining equation (4.32) we find, following Needham and Merkin [6], that provided

$$\left.\frac{dP_s}{dE}\right|_{\epsilon_0} \left\{ \epsilon_0 \left(F\beta_0' + \frac{2\epsilon_0 - 1}{\epsilon_0} \right) \right\}^{-2} > 1$$

then for any initial disturbance, $\bar{E} \to 0$ as $t \to \infty$. When this is so we expect the fluidization to be homogeneous and the solution $\bar{E} \equiv 0$ to give a reasonable first approximation. This is further confirmed by Murray [5], who used the solution $\bar{E} \equiv 0$ in the "outer" flow field when considering the passage of a single "bubble" through an otherwise homogeneously fluidized bed, his results showing encouraging agreement with experimental evidence. With this in mind, we take $\bar{E} \equiv 0$ in the fluidized region from now on throughout the paper. The equations of motion in the fluidized region then become,

$$\nabla \cdot \bar{\mathbf{u}} = 0, \tag{4.33}$$

$$\nabla \cdot \bar{\mathbf{v}} = 0, \tag{4.34}$$

$$(1-\epsilon_0)\frac{\partial \bar{\mathbf{v}}}{\partial t} = \frac{1-\epsilon_0}{F}(\bar{\mathbf{u}} - \bar{\mathbf{v}}), \tag{4.35}$$

$$\nabla \bar{P} = -\frac{1-\epsilon_0}{F}(\bar{\mathbf{u}} - \bar{\mathbf{v}}) \tag{4.36}$$

while boundary conditions (4.25), (4.29) and (4.31) become

$$\bar{u}_z = 0 \qquad \text{on} \quad z = -1, \tag{4.37}$$

$$\bar{P} = \frac{(1-\epsilon_0)\bar{\eta}}{F} \qquad \text{on} \quad z = 0, \tag{4.38}$$

$$\epsilon_0 \bar{u}_z - \bar{q}_z + (1-\epsilon_0)\bar{v}_z = 0 \qquad \text{on} \quad z = 0. \tag{4.39}$$

Equations (4.33) to (4.36) are six equations in five unknowns, but they are not all independent, since taking the divergence of equation (4.35) and using equation (4.34) gives $\nabla \cdot \bar{\mathbf{u}} = 0$, equation (4.33). Thus equation (4.33) can be removed, leaving the five independent equations. (4.34)–(4.36) in the five unknowns $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$ and $\bar{P}$. After some re-arrangement the equations to be solved in the fluidized region are

$$\nabla \cdot \bar{\mathbf{v}} = 0, \tag{4.40}$$

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} = -\frac{1}{1-\epsilon_0} \nabla \bar{P}, \tag{4.41}$$

$$\nabla \bar{P} = -\frac{1-\epsilon_0}{F}(\bar{\mathbf{u}} - \bar{\mathbf{v}}). \tag{4.42}$$

Defining $\boldsymbol{\omega}_f = \text{curl}\ \bar{\mathbf{u}}$ and $\boldsymbol{\omega}_p = \text{curl}\ \bar{\mathbf{v}}$ as the vorticity fields in the fluid and particle phases respectively, we find that by taking the curl of equations (4.41) and (4.42), $\partial \boldsymbol{\omega}_p / \partial t \equiv 0$ and $\boldsymbol{\omega}_f = \boldsymbol{\omega}_p$, i.e. the vorticity fields are independent of time, thus the rotational part of the velocity fields induced by the vorticity fields are also independent of time and by equation (4.41) have $\bar{P} = 0$. Therefore, using boundary condition (4.38), the rotational part of the velocity fields induce no fluctuation to the flatness of the free surface. The remaining part of the velocity field is irrotational, and it is only this part which disturbs the free surface. Thus since we wish to determine the possible motions of the free surface due to small perturbations about the uniform state, it is necessary only to consider the irrotational parts of the velocity fields. Therefore we may now introduce the potential fields ϕ, ψ and ψ_l, where $\bar{\mathbf{v}} = \nabla\phi$, $\bar{\mathbf{u}} = \nabla\psi$ and $\bar{\mathbf{q}} = \nabla\psi_l$. In terms of these potentials the equations in the fluidized region become,

$$\nabla^2 \phi = 0, \tag{4.43}$$

$$\bar{P} + (1-\epsilon_0)\frac{\partial \phi}{\partial t} = 0, \tag{4.44}$$

$$\psi = \phi - \frac{F}{1-\epsilon_0}\bar{P}, \tag{4.45}$$

while in the pure fluid region, equations (4.22) and (4.23) become

$$\nabla^2 \psi_l = 0, \tag{4.46}$$

$$\bar{P}_l + \frac{\partial \psi_l}{\partial t} + \epsilon_0 \frac{\partial \psi_l}{\partial z} = 0. \tag{4.47}$$

The boundary conditions may now be written in terms of ϕ, ψ and ψ_l as

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial x} = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = \sigma, \tag{4.48}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \psi}{\partial z} = 0 \quad \text{on} \quad z = -1, \tag{4.49}$$

$$\frac{\partial \psi_l}{\partial x} = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = \sigma, \tag{4.50}$$

$$\psi_l \text{ remains bounded as } z \to \infty, \tag{4.51}$$

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{F}\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0, \tag{4.52}$$

$$\frac{\partial \phi}{\partial t} = -\frac{\bar{\eta}}{F} \quad \text{on} \quad z = 0, \tag{4.53}$$

$$\epsilon_0 \frac{\partial \psi}{\partial z} - \frac{\partial \psi_l}{\partial z} + (1 - \epsilon_0)\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0. \tag{4.54}$$

We now solve in the fluidized region for ϕ, $\bar{P}$ and ψ. A separable solution for ϕ is sought in the form $\phi = X_\phi(x)Z_\phi(z)T_\phi(t)$, and the appropriate solution of equation (4.43) subject to boundary conditions (4.48), (4.49) and (4.52) is given by

$$\phi_k(x, z, t) = \cosh k(z+1) \cos kx \{A_k \sin \omega_k t + B_k \cos \omega_k t\} \tag{4.55}$$

where

$$\omega_k^2 = \frac{k}{F} \tanh k \quad \text{and} \quad k = \frac{n\pi}{\sigma},\ n = 0, 1, 2, \ldots$$

$\bar{P}$ is now determined from equation (4.44) as

$$\bar{P}_k(x, z, t) = -(1 - \epsilon_0)\omega_k \cosh k(z+1) \cos kx \{A_k \cos \omega_k t - B_k \sin \omega_k t\}, \tag{4.56}$$

while ψ follows from equation (4.45) and is given by

$$\psi_k(x, z, t) = \cosh k(z+1) \cos kx \{(A_k - F\omega_k B_k) \sin \omega_k t + (B_k + F\omega_k A_k) \cos \omega_k t\}. \tag{4.57}$$

It should be noted here that although no direct application of the boundary conditions (4.48) and (4.49) was necessary to determine ψ from equation (4.45), ψ does in fact satisfy these conditions, which may be shown directly from equations (4.44) and (4.45).

In the pure fluid region equations (4.46) and (4.47) must be solved for ψ_l and $\bar{P}_l$. Again a separable solution of equation (4.46) is sought in the form $\psi_l = X_l(x)Z_l(z)T_l(t)$. After application of the boundary conditions (4.50) and (4.51), the appropriate solution is given by

$$\psi_{lk}(x, z, t) = e^{-kz} \cos kx\ T_{lk}(t). \tag{4.58}$$

Substitution of (4.55), (4.57) and (4.58) into the boundary condition (4.54) then determines $T_k(t)$ as

$$T_{lk}(t) = \{(\epsilon_0 F\omega_k B_k - A_k)\sin\omega_k t - (\epsilon_0 F\omega_k A_k + B_k)\cos\omega_k t\}\sinh k. \tag{4.59}$$

It is now straightforward to calculate $\overline{P}_l$ from equation (4.47).

Finally, the surface elevation $\overline{\eta}$ may be determined from the remaining boundary condition (4.53) and, after use of (4.55), takes the form

$$\overline{\eta}_k(x, t) = F\omega_k \cosh k \cos kx\{B_k \sin\omega_k t - A_k\cos\omega_k t\}. \tag{4.60}$$

Thus the free-surface motions are harmonic standing waves of wave number k, where

$$k = \frac{n\pi}{\sigma},\ n = 0, 1, 2, \ldots \tag{4.61}$$

and

$$\omega(k) = \{kF^{-1}\tanh k\}^{1/2}. \tag{4.62}$$

For many fluidized beds the Froude number $F \ll 1$. Hence the period of the standing waves $T(k) = 2\pi/\omega(k) \sim 0(F/k)^{1/2} \ll 1$ for all $k = n\pi/\sigma$, $n = 1, 2, \ldots$

For arbitrary initial surface elevations, the general solutions may be written as Fourier series,

$$\overline{\eta}(x, t) = \sum_{n=0}^{\infty} F\omega_n \cosh\frac{n\pi}{\sigma}\cos\frac{n\pi x}{\sigma}\{B_n\sin\omega_n t - A_n\cos\omega_n t\},$$

$$\phi(x, z, t) = \sum_{n=0}^{\infty}\cosh\frac{n\pi}{\sigma}(z+1)\cos\frac{n\pi x}{\sigma}\{A_n\sin\omega_n t + B_n\cos\omega_n t\},$$

$$\psi(x, z, t) = \sum_{n=0}^{\infty}\cosh\frac{n\pi}{\sigma}(z+1)\cos\frac{n\pi x}{\sigma}\{(A_n - F\omega_n B_n)\sin\omega_n t$$
$$+(B_n + F\omega_n A_n)\cos\omega_n t\},$$

$$\psi_l(x, z, t) = \sum_{n=0}^{\infty} e^{-n\pi z/\sigma}\cos\frac{n\pi}{\sigma}x\{(\epsilon_0 F\omega_n B_n - A_n)\sin\omega_n t$$
$$-(\epsilon_0 F\omega_n A_n + B_n)\cos\omega_n t\}\sinh\frac{n\pi}{\sigma}.$$

Two initial conditions are then required to determine the constants A_n and B_n.

5. Fluidized bed of cylindrical cross-section

Having considered the possible two-dimensional small-amplitude surface motions in a fluidized bed of rectangular cross-section, we now examine the axisymmetric surface elevations in a cylindrical fluidized bed. Coordinates (R, θ, z) are introduced where again z measures distance vertically upwards, R measures distance radially outwards from the vertical axis of the cylinder and θ measures the angle in the horizontal plane. For axisymmetric flow, $\partial/\partial\theta \equiv 0$ and a is now the radius of the cylinder. Also, boundary conditions (4.48) and (4.50) become

$$\frac{\partial\phi}{\partial R}=\frac{\partial\psi}{\partial R}=0 \quad \text{on} \quad R=a,\ -1\leq z\leq 0, \tag{5.1}$$

$$\frac{\partial\psi_l}{\partial R}=0 \quad \text{on} \quad R=a,\ 0\leq z<\infty. \tag{5.2}$$

The appropriate solution of equation (4.43) subject to boundary conditions (5.1), (4.52) and (4.49) is

$$\phi_k(R, z, t)=\cosh k(z+1)J_0(kR)\{A_k\cos\omega_k t+B_k\sin\omega_k t\} \tag{5.3}$$

where J_0 is the Bessel function of the first kind of order zero, and

$$\omega_k=\left(\frac{k}{F}\tanh k\right)^{1/2}, \tag{5.4}$$

$$k=\frac{\mu_n}{\sigma},\ n=0, 1, 2, \ldots, \tag{5.5}$$

where μ_n are solutions of the equation $J_1(k\sigma)=0$. ψ is determined by (4.45) as

$$\psi_k(R, z, t)=\cosh k(z+1)J_0(kR)\{(A_k+F\omega_k B_k)\cos\omega_k t+(B_k-F\omega_k A_k)\sin\omega_k t\}, \tag{5.6}$$

while in the pure fluid region the appropriate solution of equation (4.46) subject to (5.2), (4.51) and (4.54) is given by

$$\psi_{lk}(R, z, t)=e^{-kz}J_0(kR)\{(\epsilon_0 F\omega_k B_k-A_k)\sin\omega_k t-(\epsilon_0 F\omega_k A_k+B_k)\cos\omega_k t\}\sinh k. \tag{5.7}$$

Finally, the surface elevation is determined from (4.53) as

$$\bar{\eta}_k(R, t)=F\omega_k\cosh kJ_0(kR)\{B_k\sin\omega_k t-A_k\cos\omega_k t\}. \tag{5.8}$$

The general solution may now be written as a Fourier-Bessel series, Watson [8], and again two initial conditions are required to determine the constants A_k, B_k.

6. Conclusions

The possible small-amplitude (small compared with depth) motions of the free surface of a homogeneously fluidized bed in which $\rho_f/\rho_s \ll 1$ have been examined. For two-dimensional flow in a fluidized bed of rectangular cross-section and axisymmetric flow in a fluidized bed of cylindrical cross-section, it has been shown that these possible surface motions are standing waves.

The surface of the rectangular bed is determined by the superposition of sinusoidal wave components with possible wave numbers $n\pi/\sigma$, $n = 0, 1, 2, \ldots$, which have corresponding period

$$T_n = 2\pi \Big/ \left\{ \frac{n\pi}{\sigma F} \tanh \frac{n\pi}{\sigma} \right\}^{1/2},$$

while the surface of the cylindrical bed is composed of components with radial dependence $J_0(\mu_n R/\sigma)$, $n = 0, 1, 2, \ldots$, and corresponding period

$$T_n = 2\pi \Big/ \left\{ \frac{\mu_n}{\sigma F} \tanh \frac{\mu_n}{\sigma} \right\}^{1/2}.$$

In terms of dimensional quantities, we find that the period is dependent only upon the depth of the undisturbed bed, h, the width (or radius) of the bed, a and g.

Furthermore, the possible motions of the free surface are exactly the same as those of the free surface of a liquid with undisturbed height h when contained in a similar tank. This correspondence becomes more clear on noticing that the equations and boundary conditions (4.43), (4.48), (4.49) and (4.52) governing the potential, ϕ, for the particle phase velocity in the fluidized region, together with condition (4.53) which determines the free surface, are identical with the linearized equations and boundary conditions determining the velocity potential and the free-surface elevation of a pure inviscid liquid of undisturbed depth h bounded by a similar vessel. Therefore a direct analogy can be made between small-amplitude surface waves on a homogeneously fluidized bed for which $\rho_f/\rho_s \ll 1$, and those on the surface of a liquid contained in a similar vessel (when surface tension is neglected). To determine the surface elevation an "inviscid" theory was used when particle and fluid phase viscosity were neglected. For gas-fluidized beds the effect of the fluid phase velocity is expected to be very small, but it is not immediately clear how the motion is affected by the neglect of the particle phase viscosity. Since this enters the linearised equations of motion as a term of the form $\nu_s \nabla^2 \mathbf{v}$, where $\nu_s = \mu_s/(1-\epsilon_0)\rho_s$, as in the equations of motion for a pure liquid, use of the above analogy can be made in determining its effect. Following Lighthill [3], we find the effect of particle phase viscosity is to cause attenuation of the surface waves through energy dissipation due to bottom friction which takes place in a boundary layer attached to the solid bottom and internal dissipation by viscous stresses acting throughout the wave, although for deep beds in which the depth is much longer than the wavelength, the attenuation is significant only over many periods, and the inviscid theory then gives a good approximation.

Therefore for fluidized beds in which $\rho_f/\rho_s \ll 1$, to leading order, small-amplitude surface waves are shown to be independent of the dynamics of the bed, and, at this order, the modelling of the free surface of a fluidized bed as that of a pure fluid is shown to be valid provided the bed is stable to small-amplitude voidage disturbances, which validates the approximation $\bar{E} \equiv 0$ through the flow field.

Acknowledgement

I would like to thank Professor C. McGreavy and Dr J.H. Merkin for help in the preparation of this paper. The author is in receipt of a SERC Research studentship.

References

[1] T.B. Anderson and R. Jackson, A fluid mechanical description of fluidized beds, *I/EC Fundament.* 6 (1967) 527–539.
[2] Gelperin and V.C. Einstein, in J.F. Davidson and D. Harrison, (eds.), *Fluidization*, Academic Press, New York (1971).
[3] M.J. Lighthill, *Waves in fluids*, Cambridge University Press (1978).
[4] J.D. Murray, On the mathematics of fluidization I, *J. Fluid Mech.* 21 (1965) 465–493.
[5] J.D. Murray, On the mathematics of fluidization II, *J. Fluid Mech.* 22 (1965) 57–80.
[6] D.J. Needham and J.H. Merkin, The propagation of a voidage disturbance in a uniformly fluidized bed, *J. Fluid Mech.* 131 (1983) 427–454.
[7] W.J. Rice and R.H. Wilhelm, Surface dynamics of fluidized beds and quality of fluidization, *A.I.Ch.E. Journal* 4, 4 (1958) 423–430.
[8] G.N. Watson, *A treatise on the theory of Bessel functions*. Cambridge University Press, 2nd edn. (1944).